

Large- N_c QCD and Weak Matrix Elements

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Abstract

I report on recent progress [1, 2] in calculating electroweak processes within the framework of QCD in the $1/N_c$ expansion.

1. Introduction

In the Standard Model, the physics of non-leptonic K -decays is described by an effective Lagrangian which is the sum of four-quark operators modulated by c -number coefficients (Wilson coefficients). This effective Lagrangian results from integrating out the fields in the Standard Model with heavy masses (Z^0 , W^\pm , t , b and c), in the presence of the strong interactions evaluated in perturbative QCD (pQCD) down to a scale μ below the mass of the charm quark M_c . The scale μ has to be large enough for the pQCD evaluation of the c -number coefficients to be valid and, therefore, it is much larger than the scale at which an effective Lagrangian description in terms of the Nambu-Goldstone degrees of freedom (K , π and η) of the spontaneous $SU(3)_L \times SU(3)_R$ symmetry breaking (S χ SB) is appropriate. Furthermore, the evaluation of the coupling constants of the low-energy effective chiral Lagrangian cannot be made within pQCD because at scales $\mu \lesssim 1$ GeV we enter a regime where S χ SB and confinement take place and the dynamics of QCD is then fully governed by non-perturbative phenomena.

The structure of the low-energy effective Lagrangian, in the absence of virtual electroweak interactions, is well-known [3]

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} f_\pi^2 \text{tr} D_\mu U D^\mu U + \cdots + L_{10} \text{tr} U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} + \cdots \quad (1)$$

Here the unitary matrix U collects the meson fields (K , π and η) and F_L , (F_R) denote field-strength tensors associated to external gauge field sources. The dots indicate other terms with the same chiral power counting $\mathcal{O}(p^4)$ as the L_{10} term and higher order terms. The important point that

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I wish to emphasize here is that *the coupling constants of this effective Lagrangian, correspond to coefficients of the Taylor expansion in powers of momenta (and quark masses), of specific QCD Green's functions of colour singlet quark-currents*. Let us consider as an example, and in the chiral limit where the light quark masses are set to zero, the two-point function ($Q^2 = -q^2$; $L^\mu = \bar{q}\gamma^\mu \frac{1}{2}(1 - \gamma_5)q$; $R^\mu = \bar{q}\gamma^\mu \frac{1}{2}(1 + \gamma_5)q$)

$$\Pi_{LR}^{\mu\nu}(q) = 2i \int d^4x e^{iq \cdot x} \langle 0 | T \left(L^\mu(x) R^\nu(0)^\dagger \right) | 0 \rangle = (q^\mu q^\nu - g^{\mu\nu} q^2) \Pi_{LR}(Q^2). \quad (2)$$

For Q^2 small, $-Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + 4L_{10} Q^2 + \mathcal{O}(Q^4)$, clearly showing the correspondence stated above.

In the presence of virtual electroweak interactions there appear new couplings in the low-energy effective Lagrangian, like e.g. the term

$$e^2 C \text{tr} \left(Q_R U Q_L U^\dagger \right) = -2e^2 C \frac{1}{f_\pi^2} (\pi^+ \pi^- + K^+ K^-) + \dots, \quad (3)$$

where $Q_R = Q_L = \text{diag}[2/3, -1/3, -1/3]$, showing that, in the presence of the electroweak interactions, the charged pion and kaon fields become massive. The basic complication in evaluating coupling constants like C in Eq. (3), which originate in loops with electroweak gauge fields, is that *they correspond to integrals over all values of the euclidean momenta of specific combinations of QCD Green's functions of colour singlet quark-currents*. In our particular example, it can be shown [4, 1] that

$$C = \frac{-1}{8\pi^2} \frac{3}{4} \int_0^\infty dQ^2 Q^2 \left(1 - \frac{1}{Q^2 + M_Z^2} \right) \Pi_{LR}(Q^2), \quad (4)$$

with Q the euclidean momentum of the virtual gauge field; the first term in the parenthesis is the well known [4] contribution from electromagnetism; the second term is the one induced by the weak neutral current [1]. It is clear that the evaluation of coupling constants of this type represents a rather formidable task. As we shall see below, it is possible, however, to proceed further within the framework of the $1/N_c$ -expansion in QCD [5].

2. Large- N_c QCD and the OPE

In the limit where the number of colours N_c becomes infinite, with $\alpha_s \times N_c$ fixed, the QCD spectrum reduces to an infinite number of zero-width mesonic resonances, and the leading large- N_c contribution to an n -point correlator is given by all the possible tree-level exchanges of these resonances in the various channels. In this limit, the analytical structure of an n -point function is very simple: the singularities in each channel consist only of a succession of *simple poles*. For example, in the case of Π_{LR} in Eq. (2),

$$-Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + \sum_A f_A^2 M_A^2 \frac{Q^2}{M_A^2 + Q^2} - \sum_V f_V^2 M_V^2 \frac{Q^2}{M_V^2 + Q^2}, \quad (5)$$

where the sums extend over all vector (V) and axial-vector (A) states. Furthermore, in the chiral limit, the operator product expansion (OPE) applied to the correlation function $\Pi_{LR}(Q^2)$ implies

$$\lim_{Q^2 \rightarrow \infty} Q^2 \Pi_{LR}(Q^2) \rightarrow 0, \quad \lim_{Q^2 \rightarrow \infty} Q^4 \Pi_{LR}(Q^2) \rightarrow 0, \quad (6)$$

and [6]

$$\lim_{Q^2 \rightarrow \infty} Q^6 \Pi_{LR}(Q^2) = -4\pi^2 \left(\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) \langle \bar{\psi}\psi \rangle^2. \quad (7)$$

The first two relations result in the two Weinberg sum rules

$$\sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = f_\pi^2 \quad \text{and} \quad \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0. \quad (8)$$

There is in fact a new set of constraints that emerge in the large- N_c limit which relate order parameters of the OPE to couplings and masses of the narrow states. In our example, we have from Eqs. (5) and (7), that

$$\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 = -4\pi^2 \left(\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) \langle \bar{\psi}\psi \rangle^2. \quad (9)$$

On the other hand, the coupling constants of the low-energy Lagrangian in the strong interaction sector are also related to couplings and masses of the narrow states of the large- N_c QCD spectrum; e.g.,

$$-4L_{10} = \sum_V f_V^2 - \sum_A f_A^2. \quad (10)$$

It is to be remarked that the convergence of the integral in Eq. (4) in the large- N_c limit is guaranteed by the two Weinberg sum rules in Eqs. (8). However, in order to obtain a numerical estimate, and in the absence of an explicit solution of QCD in the large- N_c limit, one still needs to make further approximations. Partly inspired by the phenomenological successes of “vector meson dominance” in predicting e.g., the low-energy constants of the effective chiral Lagrangian [7], we have recently proposed [8] to consider the approximation to large- N_c QCD, which restricts the hadronic spectrum to a minimal pattern, compatible with the short-distance properties of the QCD Green’s functions which govern the observable(s) one is interested in. In the channels with J^P quantum numbers 1^- and 1^+ this minimal pattern, in the cases which we have discussed so far, is the one with a spectrum which consists of a hadronic lowest energy narrow state and treats the rest of the narrow states as a large- N_c pQCD continuum; the onset of the continuum being fixed by consistency constraints from the OPE, like the absence of dimension $d = 2$ operators. We call this the *lowest meson dominance* (LMD) approximation to large- N_c QCD. The basic observation

here is that *order parameters of $S\chi SB$ in QCD have a smooth behaviour at short distances*. For example, in the case of the function Π_{LR} and, therefore, the coupling C , this is reflected by the fact that (in the chiral limit) the pQCD continuum contributions in the V -sum and the A -sum in Eq. (5) cancel each other. The evaluation of the constant C in Eq. (4) in this approximation, corresponds to a mass difference $\Delta m_\pi = 4.9$ MeV, remarkably close to the experimental result: $\Delta m_\pi|_{\text{exp.}} = 4.59$ MeV.

3. Electroweak Penguin Operators.

Within the framework discussed above, we have also shown [1] that the $K \rightarrow \pi\pi$ matrix elements of the four-quark operator

$$Q_7 = 6(\bar{s}_L\gamma^\mu d_L) \sum_{q=u,d,s} e_q(\bar{q}_R\gamma_\mu q_R), \quad (11)$$

generated by the electroweak penguin-like diagrams of the Standard Model, can be calculated to first non-trivial order in the chiral expansion and in the $1/N_c$ expansion. What is needed here is the bosonization of the operator Q_7 to next-to-leading order in the $1/N_c$ expansion. The problem turns out to be entirely analogous to the bosonization of the operator $Q_{LR} \equiv (\bar{q}_L\gamma^\mu Q_L q_L)(\bar{q}_R\gamma^\mu Q_R q_R)$ which governs the electroweak $\pi^+ - \pi^0$ mass difference discussed above. Because of the LR structure, the factorized component of Q_7 , which is leading in $1/N_c$, cannot contribute to order $\mathcal{O}(p^0)$ in the low-energy effective Lagrangian. The first $\mathcal{O}(p^0)$ contribution from this operator is next-to-leading in the $1/N_c$ expansion and is given by an integral, $[(\lambda_L^{(23)})_{ij} = \delta_{i2}\delta_{3j} \ (i, j = 1, 2, 3),]$

$$Q_7 \rightarrow -3ig_{\mu\nu} \int \frac{d^4q}{(2\pi)^4} \Pi_{LR}^{\mu\nu}(q) \text{tr} (U\lambda_L^{(23)}U^\dagger Q_R), \quad (12)$$

involving the same two-point function as in Eq. (2). Although the resulting B factors of $\Delta I = 1/2$ and $\Delta I = 3/2$ transitions are found to depend only logarithmically on the matching scale μ , their actual numerical values turn out to be rather sensitive to the precise choice of μ in the GeV region. Furthermore, because of the normalization to the vacuum saturation approximation (VSA) inherent to the (rather disgraceful) conventional definition of B -factors, there appears a spurious dependence on the light quark masses as well. In Fig. 1 we show our prediction for the ratio

$$\tilde{B}_7^{(3/2)} \equiv \frac{\langle \pi^+ | Q_7^{(3/2)} | K^+ \rangle}{\langle \pi^+ | Q_7^{(3/2)} | K^+ \rangle_0^{\text{VSA}}}, \quad (13)$$

versus the matching scale μ defined in the \overline{MS} scheme. This is the ratio considered in recent lattice QCD calculations [9]. [In fact, the lattice

definition of $\tilde{B}_7^{(3/2)}$ uses a current algebra relation between the $K \rightarrow \pi\pi$ and the $K \rightarrow \pi$ matrix elements which is only valid at order $\mathcal{O}(p^0)$ in the chiral expansion.] In Eq. (13), the matrix element in the denominator is evaluated in the chiral limit, as indicated by the subscript “0”.

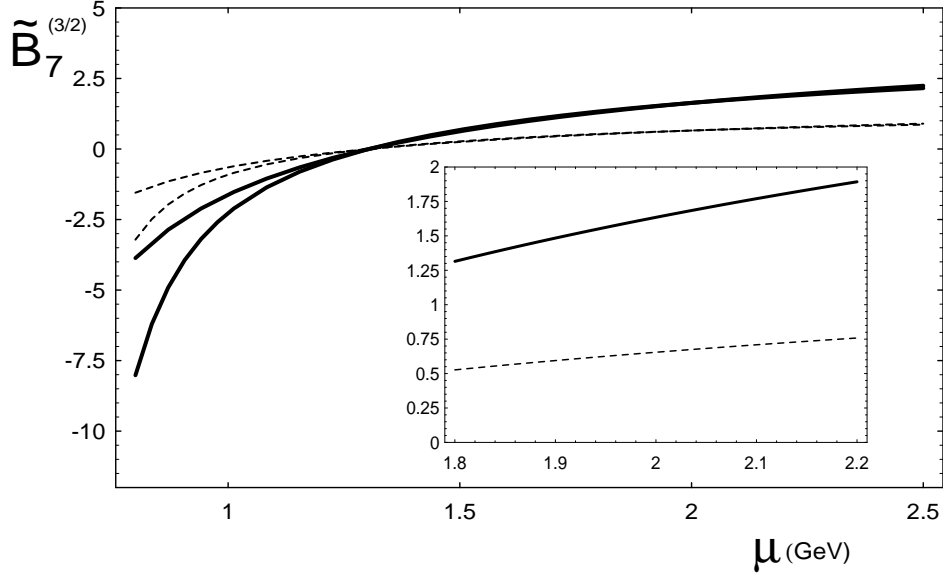


Fig. 1: The $\tilde{B}_7^{(3/2)}$ factor in Eq. (13) versus μ in GeV. Solid lines correspond to $(m_s+m_d)(2\text{ GeV}) = 158\text{ MeV}$; dashed lines to $(m_s+m_d)(2\text{ GeV}) = 100\text{ MeV}$.

4. Decay of Pseudoscalars into Lepton Pairs.

The processes $\pi \rightarrow e^+e^-$ and $\eta \rightarrow l^+l^-$ ($l = e, \mu$) are dominated by the exchange of two virtual photons. It is then useful to consider the ratios ($P = \pi^0, \eta$)

$$R(P \rightarrow \ell^+\ell^-) = \frac{Br(P \rightarrow \ell^+\ell^-)}{Br(P \rightarrow \gamma\gamma)} = 2 \left(\frac{\alpha m_\ell}{\pi M_P} \right)^2 \beta_\ell(M_P^2) |\mathcal{A}(M_P^2)|^2, \quad (14)$$

with $\beta_\ell(s) = \sqrt{1 - 4m_\ell^2/s}$. To lowest order in the chiral expansion, the unknown dynamics in the amplitude $\mathcal{A}(M_P^2)$ depends entirely on a low-energy coupling constant χ . We have recently shown [2] that this constant can be expressed as an integral over the three-point function

$$\begin{aligned} \int d^4x \int d^4y e^{iq_1 \cdot x} e^{iq_2 \cdot y} < 0 | T \{ j_\mu^{\text{em}}(x) j_\nu^{\text{em}}(y) P^3(0) \} | 0 > \\ = \frac{2}{3} \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{H}(q_1^2, q_2^2, (q_1 + q_2)^2), \end{aligned} \quad (15)$$

involving the electromagnetic current j_μ^{em} and the density current $P^3 = \frac{1}{2}(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d)$. More precisely, (d =space-time dimension,)

$$\frac{\chi(\mu)}{32\pi^4} \frac{\langle \bar{\psi}\psi \rangle}{F_\pi^2} = - \left(1 - \frac{1}{d} \right) \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2} \right)^2 \times$$

$$\lim_{(p'-p)^2 \rightarrow 0} (p' - p)^2 \left[\mathcal{H}(q^2, q^2, (p' - p)^2) - \mathcal{H}(0, 0, (p' - p)^2) \right]. \quad (16)$$

The evaluation of this coupling in the LMD approximation to large- N_c QCD which we have discussed above, leads to the result $\chi^{\text{LMD}}(\mu = M_V) = 2.2 \pm 0.9$. The corresponding branching ratios are shown in Table 1.

Table 1: Ratios $R(P \rightarrow \ell^+ \ell^-)$ in Eq. (14) obtained within the LMD approximation to large- N_c QCD and the comparison with available experimental results.

R	LMD	Experiment
$R(\pi^0 \rightarrow e^+ e^-) \times 10^8$	6.2 ± 0.3	7.13 ± 0.55 [10]
$R(\eta \rightarrow \mu^+ \mu^-) \times 10^5$	1.4 ± 0.2	1.48 ± 0.22 [11]
$R(\eta \rightarrow e^+ e^-) \times 10^8$	1.15 ± 0.05	?

It was shown in ref. [12] that, when evaluated within the chiral $U(3)$ framework and in the $1/N_c$ expansion, the $|\Delta S| = 1$ $K_L^0 \rightarrow \ell^+ \ell^-$ transitions can also be described by an expression like in Eq. (14) with an effective constant $\chi_{K_L^0}$ containing an additional piece from the short-distance contributions [13]. The most accurate experimental determination [14] gives: $Br(K_L^0 \rightarrow \mu^+ \mu^-) = (7.18 \pm 0.17) \times 10^{-9}$. In the framework of the $1/N_c$ expansion and using the experimental branching ratio [11] $Br(K_L^0 \rightarrow \gamma\gamma) = (5.92 \pm 0.15) \times 10^{-4}$, this leads to a unique solution for an *effective* $\chi_{K_L^0} = 5.17 \pm 1.13$. Furthermore, following Ref. [12], $\chi_{K_L^0} = \chi - \mathcal{N} \delta\chi_{SD}$ where $\mathcal{N} = (3.6/g_8 c_{\text{red}})$ normalizes the $K_L^0 \rightarrow \gamma\gamma$ amplitude. The coupling g_8 governs the $\Delta I = 1/2$ rule, the constant c_{red} is defined in Ref. [12] and $\delta\chi_{SD}^{\text{Standard}} = (+1.8 \pm 0.6)$ is the short-distance contribution in the Standard Model [13]. Therefore, a test of the *short-distance* contribution to this process completely hinges on our understanding of the *long-distance* constant \mathcal{N} and therefore of the $\Delta I = 1/2$ rule in the $1/N_c$ expansion. Moreover, c_{red} is regrettably very unstable in the chiral and large- N_c limits, a behaviour that surely points towards the need to have higher order corrections under control. The analysis of Ref. [12] uses $c_{\text{red}} \simeq +1$ and $g_8 \simeq 3.6$, where these numbers are obtained phenomenologically by requiring agreement with the two-photon decay of K_L^0, π^0, η and η' as well as $K \rightarrow 2\pi, 3\pi$. Should we use these values of c_{red} and g_8 with our result $\chi^{\text{LMD}}(\mu = M_V) = 2.2 \pm 0.9$ we would obtain $\chi_{K_L^0} = 0.4 \pm 1.1$, corresponding to a ratio $R(K_L^0 \rightarrow \mu^+ \mu^-) = (2.24 \pm 0.41) \times 10^{-5}$ which is 2.5σ above the experimental value $R(K_L^0 \rightarrow \mu^+ \mu^-) = (1.21 \pm 0.04) \times 10^{-5}$.

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